A note on Strichartz estimates for Airy equation and its application (Harmonic Analysis and Nonlinear Partial Differential Equations)

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By

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§ 1. Strichartz estimates

The main purpose of the survey note is to review recent progress on the Strichartz estimates for the Airy equation:

$$\begin{cases} \partial_t u + \partial_x^3 u = 0, & t, x \in \mathbb{R}, \\ u(0,x) = f(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown function and $f : \mathbb{R} \to \mathbb{R}$ is a given function. As with the Schrödinger equation, the Strichartz estimate for (1.1) have been studied by many authors. Let us review some of them. Throughout this survey, the operator $|\partial_x|^s = (-\partial_x^2)^{s/2}$ denotes the Riesz potential of order $-s$. For $1 \leqslant p, q \leqslant \infty$, let us define a space-time norm $\|f\|_{L^p_x L^q_t} = \|\|f(\cdot,x)\|_{L^q_t(\mathbb{R})}\|_{L^p_x(\mathbb{R})}$.

Strichartz' estimate for (1.1) is derived by Kenig, Ponce and Vega [13]:

Theorem 1.1 (Strichartz' estimate [13]). *Let (p,q) be a pair satisfying*

$$0 \leqslant \frac{1}{p} \leqslant \frac{1}{4}, \qquad \frac{2}{p} + \frac{1}{q} = \frac{1}{2}.$$

Let $e^{-t\partial_x^3} f$ be a solution to (1.1). Then, there exists a positive constant C depends only on p and q such that the inequality

$$\||\partial_x|^s e^{-t\partial_x^3} f\|_{L^p_x(\mathbb{R};L^q_t(\mathbb{R}))} \leqslant C \|f\|_{L^2} \tag{1.2}$$

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holds for any $f \in L^2$, where s is given by

$$s = -\frac{1}{p} + \frac{2}{q}.$$

To obtain Theorem 1.1, they derived two important inequalities, one is the Kato smoothing effect ((1.2) with $(p,q) = (\infty, 2)$) which is a variant of the local smoothing effect for the Airy equation discovered by T.Kato [11], and the other is the Kenig-Ruiz estimate ((1.2) with $(p,q) = (4,\infty)$). Note that the Kato smoothing effect tell us that the solutions to the Airy equation (1.1) have gain of spatial regularity of order one in $L^\infty_x(\mathbb{R}; L^2_t(\mathbb{R}))$. These two estimates correspond to the two endpoint cases. Hence, the other case follows by interpolation.

By using the Strichartz estimate (1.2), they [14] succeeded to prove the local well-posedness for the Cauchy problem of the generalized Korteweg-de Vries (gKdV) equation:

$$\text{(gKdV)} \qquad \begin{cases} \partial_t u + \partial_x^3 u = \mu \partial_x (|u|^{2\alpha} u), & t, x \in \mathbb{R}, \\ u(t_0, x) = u_0(x), & x \in \mathbb{R} \end{cases}$$

in low order Sobolev space $H^s(\mathbb{R})$, where $t_0 \in \mathbb{R}$, $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown function, $u_0 : \mathbb{R} \to \mathbb{R}$ is a given data, and $\alpha > 0$, $\mu \in \mathbb{R}\backslash\{0\}$ are constants. Indeed, they made use of the Kato smoothing effect for the Airy group to compensate a loss of derivatives caused by the nonlinear term and could apply the Banach fixed point theorem to the corresponding integral equation.

Later, Grünrock [10] and the authors [20] extended the Strichartz estimate for (1.1) to the hat-Lebesgue space. More precisely, we obtained the following estimate:

Theorem 1.2 (Generalized Strichartz' estimate [10, 20]). *Let (p,q) be a pair satisfying either $(p,q) = (\infty, 2), (4.\infty)$ or*

$$0 \leqslant \frac{1}{p} < \frac{1}{4}, \qquad 0 \leqslant \frac{1}{q} < \frac{1}{2} - \frac{1}{p}.$$

Then, there exists a positive constant C depends only on p and q such that the inequality

$$\| |\partial_x|^s e^{-t\partial_x^3} f \|_{L^p_x(\mathbb{R}; L^q_t(\mathbb{R}))} \leqslant C \|f\|_{\hat{L}^\alpha} \tag{1.3}$$

holds for any $f \in \hat{L}^\alpha$, where α and s are given by

$$\frac{1}{\alpha} = \frac{2}{p} + \frac{1}{q}, \quad s = -\frac{1}{p} + \frac{2}{q}.$$

Here the space $\hat{L}^\alpha$ is defined for $1 \leqslant \alpha \leqslant \infty$ by

$$\hat{L}^\alpha = \hat{L}^\alpha(\mathbb{R}) := \{ f \in \mathcal{S}'(\mathbb{R}) | \, \|f\|_{\hat{L}^\alpha} = \|\hat{f}\|_{L^{\alpha'}} < \infty \},$$

where $\hat{f}$ stands for the Fourier transform of f in x, and α' denotes the Hölder conjugate of α.

The key ingredient of the proof for Theorem 1.2 is the Stein-Tomas estimates for the Airy equation:

$$\left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} f \right\|_{L^{3\alpha}_{t,x}(\mathbb{R}\times\mathbb{R})} \leqslant C \|f\|_{\hat{L}^\alpha}, \tag{1.4}$$

where $4/3 < \alpha \leqslant \infty$. The Stein-Tomas estimate is classically well-known for the Schrödinger equation [9, 30] from the point of view of restriction estimate of Fourier transform. In [20], we gave a simple proof of (1.4) which is based on the reduction to bilinear from and Hausdorff-Young inequality, see [20, Lemma 2.2] for the detail.

Combining the generalized Strichartz' estimate (Theorem 1.2) and the Fourier restriction norm, Grünrock [10] has shown the local well-posedness of the modified KdV equation (i.e., (gKdV) with $\alpha = 1$) in the framework of the hat-Lebesgue type space $\langle\partial_x\rangle^{-s}\hat{L}^\beta$. In [20], the author obtained global well-posedness for small data for (gKdV) in the scaling critical hat-Lebesgue space $\hat{L}^\alpha$ with $8/5 < \alpha < 10/3$ by using the generalized Strichartz' estimate. We discuss this application in Section 4 (see Theorem 4.3, below).

§ 2. Refinement of Strichartz' estimates

In this section we consider the refinement of the Strichartz estimates for the Airy equation (1.1) in the previous section, in terms of Morrey-type spaces.

As far as the authors know, the refinement of the Strichartz estimate in this direction first appeared in [3] in a context of Schrödinger equation. Besides its own interests, the refined estimate has been studied because of its application. In [4], Bourgain use it to show a concentration phenomenon of blow-up solutions for the two dimensional mass-critical nonlinear Schrödinger equation. After Bourgain, the refinement of Strichartz' estimates are used by several authors, see [1, 5, 23, 25, 26] for instance.

As for the Airy equation (1.1), Kenig, Ponce and Vega [15] obtained the following refined estimate:

Theorem 2.1 ([15]). *Let $1 \leqslant \gamma < \infty$. Then there exists a positive constant C depending only on γ such that the inequality*

$$\left\| |\partial_x|^{\frac{1}{6}} e^{-t\partial_x^3} f \right\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})} \leqslant C \left(\sup_{I\subset\mathbb{R}} |I|^{\frac{1}{\gamma}-\frac{1}{2}} \|\hat{f}\|_{L^{\gamma'}(I)} \right)^{\frac{1}{3}} \|f\|_{L^2}^{\frac{2}{3}} \tag{2.1}$$

holds.

Remark that the right hand side is bounded by $\|f\|_{L^2}$ up to a constant. This kind of refinement is useful because if we know the left hand side of (2.1) is bounded from below and $\|f\|_{L^2}$ is bounded from above, in some way, then we can find an interval I_0 on which the Fourier transform of f concentrate in such a sense that $\|\hat{f}\|_{L^{\gamma'}(I_0)} \gtrsim |I_0|^{-\frac{1}{\gamma}+\frac{1}{2}}$, where the implicit constant depends on the two bounds. In [15], they used the above refined estimate (2.1) to study a concentration of blow-up solution for the mass-critical generalized KdV equation. See also Shao [27] for the alternative proof of the refined Stein-Tomas estimate (2.1). In this way, one main motivation to study the refinement for the Strichartz estimates lies in its application to nonlinear problems.

One specific application in our mind is the existence of a special non-scattering solution, which is minimal in a suitable sense, to the generalized KdV equation (gKdV), by using the concentration compactness argument by Kenig-Merle [12]. Let us quickly review several results in this direction. Killip, Kwon, Shao and Vişan [16] constructed a minimal blow-up solution to (gKdV) with the mass critical nonlinearity in the framework of L^2. Dodson [7] proved the global well-posedness and scattering in L^2 for (gKdV) with the defocusing (i.e. $\mu > 0$) and mass critical nonlinearity. Farah, Linares, Pastor and Visciglia [8] has shown the global well-posedness and scattering in H^1 for (gKdV) with the defocusing and mass supercritical nonlinearity. As for (gKdV), the mass critical and supercritical cases $\alpha \geqslant 2$ are most extensively studied in this direction.

We shall turn on existence of a minimal solution for (gKdV) in the mass subcritical case $\alpha < 2$. In view of scaling, the choice of the function space of solutions is a first obstacle to attack this problem. As explained in [21, Section 1], a good well-posedness theory (small data scattering, stability theorem, etc) and a decoupling (in)equality play a central role in the concentration compactness argument. Therefore, it is natural to consider the problem in the framework of the scaling critical function space.

Let us first consider the scaling critical homogeneous Sobolev space $\dot{H}^{s_\alpha}(\mathbb{R})$, where $s_\alpha := 1/2 - 1/\alpha$. The feature of the mass subcritical case $\alpha < 2$ is the critical regularity s_α is negative. This prevents us from evaluating the nonlinear term via the Leibniz rule for the fractional derivatives. We would remark that, as for the quartic nonlinearity $\mu\partial_x(u^4)$, the global well-posedness for small data in $\dot{H}^{s_{3/2}}$ is proved by Tao [28] by using the Fourier restriction norm despite of negative critical regularity $s_{3/2} = -1/6$ (see also Koch and Marzuola [17]).

Next we consider this problem in the scaling critical hat-Lebesgue space $\hat{L}^\alpha$. As mentioned before, the authors [20] obtained global well-posedness for small data for (gKdV) in the space $\hat{L}^\alpha$ with $8/5 < \alpha < 10/3$ by using the generalized Strichartz' estimate (1.3). In [21], we proved existence of the minimal non-scattering solution in $\hat{L}^\alpha$ by introducing refinement of (1.3) for $\alpha \neq 2$. However, we imposed several technical assumptions due to the lack of the decoupling (in)equality in $\hat{L}^\alpha$ for $\alpha \neq 2$. We discuss details in Section 4.

Let us move on to the precise statement of the refinements. To this end, we introduce a generalized hat-Morrey space.

Definition 2.2 (Generalized hat-Morrey space)**.** For $j,k\in\mathbb{Z}$, let $\tau_k^j=[k2^{-j},(k+1)2^{-j})$ be a dyadic interval. For $1<\beta<\gamma\leqslant\infty$ and $\beta'<\delta\leqslant\infty$, we define a hat-Morrey norm by

$$\|f\|_{\hat{M}^{\beta}_{\gamma,\delta}}=\left\|\,|\tau_k^j|^{\frac{1}{\gamma}-\frac{1}{\beta}}\|\hat{f}\|_{L^{\gamma'}(\tau_k^j)}\right\|_{\ell^{\delta}_{j,k}}.$$

Banach space $\hat{M}^{\beta}_{\gamma,\delta}$ is defined as set of tempered distributions of which above norm is finite.

Remark 1. The case $\delta=\infty$ corresponds to the hat-Morrey space, the usual Morrey space in the Fourier side. And, so the above is a generalization because we allow $\delta<\infty$. Remark that we have $\cup_{k\in\mathbb{Z}}\tau_k^j=\mathbb{R}$ for each fixed $j\in\mathbb{Z}$ and that summation is also taken with respect to j. Nevertheless, one sees that the norm is finite for a class of functions. In particular, we have the following embedding $\hat{L}^{\beta}\hookrightarrow\hat{M}^{\beta}_{\gamma,\delta}$ as long as $1\leqslant\gamma'<\beta'<\delta\leqslant\infty$, see [21, Proposition A.1] for the proof.

Our first refinement is the following:

Theorem 2.3 (Refined Strichartz' estimate (diagonal case) [21])**.** *Suppose that $4/3<\alpha<\infty$. Then there exists a positive constant C depending only on α such that the inequality*

$$\left\|\,|\partial_x|^{\frac{1}{3\alpha}}e^{-t\partial_x^3}f\right\|_{L^{3\alpha}_{t,x}(\mathbb{R}\times\mathbb{R})}\leqslant C\,\|f\|_{\hat{M}^{\alpha}_{\frac{3}{2}\alpha,2(\frac{3}{2}\alpha)'}}\tag{2.2}$$

holds for any $f\in\hat{M}^{\alpha}_{\frac{3}{2}\alpha,2(\frac{3}{2}\alpha)'}$.

A novelty of the above estimate is that we allow $\alpha\neq2$. Note that, in all above previous studies, the refinements were restricted to the case $\alpha=2$. By using (2.2), we are able to prove the existence of a minimal non-scattering solution for the *mass subcritical* ($\alpha<2$) generalized KdV equation in the $\hat{L}^{\alpha}$-framework.

Remark 2. Recall the embedding $\hat{L}^{\beta}\hookrightarrow\hat{M}^{\beta}_{\gamma,\delta}$ for $1\leqslant\gamma'<\beta'<\delta\leqslant\infty$. Hence Theorem 2.3 is an improvement of the Stein-Tomas estimate (1.4).

In [22], we further extended the refinement to the *non-diagonal* case $p\neq q$. Note again that the refinements were restricted to the *diagonal* case $p=q$ in previous studies. A similar refinement in $\hat{M}^{\beta}_{\gamma,\delta}$-framework for the Schrödinger equation was done by the first author [19], including its application to existence of a minimal non-scattering solution for the mass-subcritical nonlinear Schrödinger equation. However, the refinement is still restricted to the diagonal case $p=q$.

Theorem 2.4 (Refined Strichartz' estimate (nondiagonal case) [22]). *Take* $\sigma \in (0,1/4)$*. Let* (p,q) *satisfy*

$$0 \leqslant \frac{1}{p} \leqslant \frac{1}{4} - \sigma, \qquad \frac{1}{q} \leqslant \frac{1}{2} - \frac{1}{p} - \sigma.$$

Define α *and* s *by*

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{\alpha}, \qquad s = -\frac{1}{p} + \frac{2}{q}.$$

Further, we define β*,* γ*, and* δ *by*

$$\frac{1}{\beta} = \frac{1}{\alpha} + \sigma, \qquad \frac{1}{\gamma} = \begin{cases} \frac{1}{\beta} - \frac{1}{p} & \text{if } \frac{1}{q} \geqslant \frac{1}{p} + \sigma, \\ \frac{1}{\beta} - \frac{1}{q} + \sigma & \text{if } \frac{1}{q} < \frac{1}{p} + \sigma, \end{cases} \qquad \frac{1}{\delta} = \frac{1}{2} - \frac{1}{\max(p,q)}.$$

Then, there exists a positive constant C *depending on* p, q, σ *such that the inequality*

$$\left\| |\partial_x|^s e^{-t\partial_x^3} f \right\|_{L^p_x(\mathbb{R}; L^q_t(\mathbb{R}))} \leqslant C \| |\partial_x|^\sigma f \|_{\hat{M}^\beta_{\gamma,\delta}} \tag{2.3}$$

holds for any $f \in |\partial_x|^{-\sigma} \hat{M}^\beta_{\gamma,\delta}$.

Question. For the diagonal case $p = q$, the inequality (2.3) holds for $\sigma = 0$ (see Theorem 2.3). Is it possible to choose $\sigma = 0$ also in the nondiagnal refinements? See Remark 3 for the reason why we need $\sigma > 0$ for now.

Note that the refined Strichartz' estimate (Theorem 2.4) for nondiagonal case enables us to prove the well-posedness of (gKdV) and existence of a minimal non-scattering solution for (gKdV) with the mass-subcritical nonlinearity in $\hat{M}^\beta_{\gamma,\delta}$-framework, as we see in Section 4.

The diagonal case $p = q$ can be handled by the bilinear technique and the Hausdorff-Young inequality as in [15, 21, 27]. However, this approach does not work well in the non-diagonal case. Furthermore, due to lack of an interpolation between the Morrey space and the Lebesgue space, the desired estimate does not follow by a simple interpolation. To overcome those difficulties, we take another approach which is based on [1, 18, 21, 31]. We outline the proof in Section 3.

§ 3. Outline of the proof of Theorem 2.4.

In this section we give the outline of the proof of Theorem 2.4. The proof is based on the argument by [1, 18, 21, 31].

Step 1: Reduction to bilinear form. To show the inequality (2.3), we first reduce the linear form into a *bilinear* form:

$$\||\partial_x|^s e^{-t\partial_x^3} f\|^2_{L^p_x L^q_t} = \|||\partial_x|^s e^{-t\partial_x^3} f|^2\|_{L^{\frac{p}{2}}_x L^{\frac{q}{2}}_t}. \tag{3.1}$$

Since f is a real valued function, we have

$$\begin{aligned}||\partial_x|^s e^{-t\partial_x^3} f|^2 &= \frac{1}{\pi} Re \int_0^\infty \int_0^\infty e^{ix(\xi+\eta)+it(\xi^3+\eta^3)} |\xi\eta|^s \hat{f}(\xi)\hat{f}(\eta) d\xi d\eta \\ &\quad + \frac{1}{\pi} Re \int_0^\infty \int_0^\infty e^{ix(\xi-\eta)+it(\xi^3-\eta^3)} |\xi\eta|^s \hat{f}(\xi)\overline{\hat{f}(\eta)} d\xi d\eta.\end{aligned}$$

Step 2: Whitney decomposition. The stationary points for the above oscillatory integrals lie on the diagonal line $\xi = \eta$. Therefore, to evaluate those oscillatory integrals, we introduce a *Whitney decomposition* adapted to the diagonal line. Let $\mathcal{D}_+ = \{[k2^{-j}, (k+1)2^{-j}) | j \in \mathbb{Z}, 0 \leqslant k \in \mathbb{Z}\}$. For τ_k^j, $\tau_\ell^j \in \mathcal{D}_+$, we define a binary relation

$$\tau_k^j \sim \tau_\ell^j \;\Leftrightarrow\; \begin{cases} \ell - k = -2, 2, 3 & \text{if } k \text{ is even}, \\ \ell - k = -3, -2, 2 & \text{if } k \text{ is odd}. \end{cases} \tag{3.2}$$

Then, we have $(\mathbb{R}_+ \times \mathbb{R}_+) \backslash \{(\xi,\xi) | \xi \geqslant 0\} = \bigcup \{\tau_k^j \times \tau_\ell^j | \tau_k^j \in \mathcal{D}_+, \tau_\ell^j : \tau_\ell^j \sim \tau_k^j\}$. The Whitney decomposition gives us

$$\begin{aligned} &||\partial_x|^s e^{-t\partial_x^3} f|^2 \\ &= \frac{1}{\pi} \sum_{\tau_k^j \in \mathcal{D}_+} \sum_{\tau_\ell^j : \tau_\ell^j \sim \tau_k^j} Re \int_{\tau_k^j} \int_{\tau_\ell^j} e^{ix(\xi+\eta)+it(\xi^3+\eta^3)} |\xi\eta|^s \hat{f}(\xi)\hat{f}(\eta) d\xi d\eta \\ &\quad + \frac{1}{\pi} \sum_{\tau_k^j \in \mathcal{D}_+} \sum_{\tau_\ell^j : \tau_\ell^j \sim \tau_k^j} Re \int_{\tau_k^j} \int_{\tau_\ell^j} e^{ix(\xi-\eta)+it(\xi^3-\eta^3)} |\xi\eta|^s \hat{f}(\xi)\overline{\hat{f}(\eta)} d\xi d\eta \\ &= 2Re \sum_{\tau_k^j \in \mathcal{D}_+} \sum_{\tau_\ell^j : \tau_\ell^j \sim \tau_k^j} |\partial_x|^s e^{-t\partial_x^3} f_{\tau_k^j} |\partial_x|^s e^{-t\partial_x^3} f_{\tau_\ell^j} \\ &\quad + 2Re \sum_{\tau_k^j \in \mathcal{D}_+} \sum_{\tau_\ell^j : \tau_\ell^j \sim \tau_k^j} |\partial_x|^s e^{-t\partial_x^3} f_{\tau_k^j} \overline{|\partial_x|^s e^{-t\partial_x^3} f_{\tau_\ell^j}} \\ &=: I_1 + I_2, \end{aligned} \tag{3.3}$$

where $\hat{f}_I(\xi) = \mathbf{1}_I(\xi)\hat{f}(\xi)$. A simple calculation leads

$$\begin{aligned} &\operatorname{supp} \mathcal{F}_{t,x}[|\partial_x|^s e^{-t\partial_x^3} f_{\tau_k^j} |\partial_x|^s e^{-t\partial_x^3} f_{\tau_\ell^j}](\tau,\xi) \\ &\subset \{(\xi_1^3 + \xi_2^3, \xi_1 + \xi_2) | \xi_1 \in \tau_k^j, \xi_2 \in \tau_\ell^j\} \\ &\subset A_{j,k,\ell}, \end{aligned}$$

where $A_{j,k,\ell}$ is given by

$$A_{j,k,\ell} = \left\{ (\tau,\xi) \;\middle|\; \frac{k+\ell}{2^j} \leqslant \xi \leqslant \frac{k+\ell+2}{2^j},\ \tau \text{ satisfies (3.5)} \right\} \tag{3.4}$$

with

$$\begin{cases} \frac{3}{4}\frac{(k-\ell-1)^2}{2^{2j}}\xi \leqslant \tau - \frac{1}{4}\xi^3 \leqslant \frac{3}{4}\frac{(k-\ell+1)^2}{2^{2j}}\xi & \text{if } \ell - k = -3,-2, \\ \frac{3}{4}\frac{(k-\ell+1)^2}{2^{2j}}\xi \leqslant \tau - \frac{1}{4}\xi^3 \leqslant \frac{3}{4}\frac{(k-\ell-1)^2}{2^{2j}}\xi & \text{if } \ell - k = 2,3. \end{cases} \tag{3.5}$$

In a similar way, we see

$$\begin{aligned} &\operatorname{supp} \mathcal{F}_{t,x}[|\partial_x|^s e^{-t\partial_x^3} f_{\tau_k^j} \overline{|\partial_x|^s e^{-t\partial_x^3} f_{\tau_\ell^j}}](\tau,\xi) \\ &\subset \{(\xi_1^3+\xi_2^3, \xi_1+\xi_2) | \xi_1 \in \tau_k^j, \xi_2 \in \tau_{-\ell-1}^j\} \\ &\subset B_{j,k,\ell}, \end{aligned}$$

where $B_{j,k,\ell}$ is given by

$$B_{j,k,\ell} = \left\{ (\tau,\xi) \;\middle|\; \frac{k-\ell-1}{2^j} \leqslant \xi \leqslant \frac{k-\ell+1}{2^j},\ \tau \text{ satisfies (3.7)} \right\}, \tag{3.6}$$

with

$$\begin{cases} \frac{3}{4}\frac{(k+\ell)^2}{2^{2j}}\xi \leqslant \tau - \frac{1}{4}\xi^3 \leqslant \frac{3}{4}\frac{(k+\ell+2)^2}{2^{2j}}\xi & \text{if } \ell - k = -3,-2, \\ \frac{3}{4}\frac{(k+\ell+2)^2}{2^{2j}}\xi \leqslant \tau - \frac{1}{4}\xi^3 \leqslant \frac{3}{4}\frac{(k+\ell)^2}{2^{2j}}\xi & \text{if } \ell - k = 2,3. \end{cases} \tag{3.7}$$

Step 3: Almost orthogonality. To evaluate the summation with respect to j,k in I_1 and I_2, we apply the interpolation for the linear operator (3.13). In this step, we have to add a small *margin* in space and time directions for $A_{j,k,\ell}$ and $B_{j,k,\ell}$. Obviously, these margins produce many doublings which may disturb orthogonality of the forms. However, if the margin is putted so nicely that the resulting doublings are acceptable then we obtain the desired estimate. The property is summarized as an *almost orthogonal property* of the Fourier supports of the forms. In the Schrödinger case, we can put a margin so that the almost orthogonal property is valid (see [1]). However, in the Airy case, the cubic dispersion makes the situation much worse and it seems that there is no way to put such a margin for smooth cutoff. An idea here is to put the margin *only in time direction*. Although this requires an unpleasant restriction $\sigma > 0$ in Theorem 2.4, we recover the almost orthogonal property, which is a key ingredient of the proof of Theorem 2.4.

Let us introduce two preliminary estimates associated with the sets $A_{j,k,\ell}$ and $B_{j,k,\ell}$. For a closed domain $R \subset \mathbb{R}^2$ and $\lambda > 0$, we define

$$R_{+\lambda} = \{(\tau + \tau', \xi) | (\tau, \xi) \in R, \ -\lambda \leqslant \tau' \leqslant \lambda\}.$$

The set $R_{+\lambda}$ is an enlargement of R in τ-direction. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a nonnegative function such that $\operatorname{supp} \varphi \subset [-1, 1]$ and $\int_{-1}^1 \varphi(x) dx = 1$. Define a cut-off function

$$\psi_{R,\lambda}(\tau, \xi) := \left[\frac{2}{\lambda} \varphi \left(\frac{2}{\lambda} (\cdot) \right) *_\tau \mathbf{1}_{R_{+\frac{\lambda}{2}}}(\cdot, \xi) \right] (\tau).$$

Note that $\psi_{R,\lambda}$ is smooth function with respect to τ variable. Furthermore, $\psi_{R,\lambda}$ satisfies $0 \leqslant \psi_{R,\lambda} \leqslant 1$, $\psi_{R,\lambda} \equiv 1$ on R, and $\operatorname{supp} \psi_{R,\lambda} \subset R_{+\lambda}$. We define a Fourier multiplier $P_{R,\lambda}$ by

$$\begin{aligned} (P_{R,\lambda} f)(t, x) &:= \mathcal{F}^{-1}_{\tau,\xi} [\psi_{R,\lambda} \mathcal{F}_{t,x} f](t, x) \\ &= \left(\mathcal{F}^{-1}_\tau [\varphi] \left(\frac{\lambda}{2} t \right) \mathcal{F}^{-1}_{\tau,\xi} [\mathbf{1}_{R_{+\frac{\lambda}{2}}}] * f \right) (t, x). \end{aligned} \tag{3.8}$$

Let $\Lambda = \{(j, k, \ell) \in \mathbb{Z} \times \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} | \ell - k = -3, -2, 2, 3\}$. For $(j, k, \ell) \in \Lambda$, we let two families of sets $\{A_{j,k,\ell}\}$ and $\{B_{j,k,\ell}\}$ be as in (3.4) and (3.6), respectively. We further introduce

$$\widetilde{A}_{j,k,\ell} = (A_{j,k,\ell})_{+\frac{k}{100 \times 2^{3j}}}, \quad \widetilde{B}_{j,k,\ell} = (B_{j,k,\ell})_{+\frac{k}{100 \times 2^{3j}}}. \tag{3.9}$$

The following finite doubling properties of the two families $\{\tilde{A}_{j,k,\ell}\}$ and $\{\tilde{B}_{j,k,\ell}\}$ play a crucial role in the proof of Theorem 2.4.

Proposition 3.1 (Almost orthogonality). *Let $X = A$ or B. Then the inequality*

$$\sum_{(j,k,\ell) \in \Lambda} \mathbf{1}_{\widetilde{X}_{j,k,\ell}}(\tau, \xi) \leqslant 12 \tag{3.10}$$

holds for almost all $(\tau, \xi) \in \mathbb{R}^2$, where $\Lambda = \{(j, k, \ell) \in \mathbb{Z} \times \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} | \ell - k = -3, -2, 2, 3\}$.

The proof of Proposition 3.1 follows from elementary (but technical) algebraic inequalities. See [22, Proposition 2.1] for the detail of the proof.

Remark 3. For closed domain $R \subset \mathbb{R}^2$ and $\lambda > 0$, we define

$$R'_{+\lambda} = \{(\tau + \tau', \xi + \xi') | (\tau, \xi) \in R, \ -\lambda \leqslant \tau' \leqslant \lambda, -\lambda \leqslant \xi' \leqslant \lambda\},$$

which is an enlargement both in τ- and ξ-directions. Further, we define $\tilde{A'}_{j,k,\ell}$ $\tilde{B'}_{j,k,\ell}$ by

$$\widetilde{A'}_{j,k,\ell} = (A_{j,k,\ell})'_{+\frac{k}{100 \times 2^{3j}}}, \quad \widetilde{B'}_{j,k,\ell} = (B_{j,k,\ell})'_{+\frac{k}{100 \times 2^{3j}}}.$$

If we are able to show the almost orthogonality properties of the two families $\{\tilde{A}'_{j,k,\ell}\}$ and $\{\tilde{B}'_{j,k,\ell}\}$, then we will be able to obtain Theorem 2.4 with $\sigma = 0$ by means of [29, Lemma 6.1], in essentially the same spirit as in [1]. In the Schrödinger case, it is possible, as shown in [1]. The difficulty comes from the fact that the dispersion is cubic.

Next we show the boundedness for the Fourier multipliers $P_{\tilde{A}_{j,k,\ell}}$ and $P_{\tilde{B}_{j,k,\ell}}$ defined by

$$P_{\tilde{X}_{j,k,\ell}} := P_{X_{j,k,\ell},\frac{k}{100\times 2^{3j}}} \qquad \text{for } X = A, B,$$

where $P_{X_{j,k,\ell},\frac{k}{100\times 2^{3j}}}$ is given by (3.8).

Proposition 3.2 (Boundedness for multiplier). *Let $X = A$ or B. Let $\sigma > 0$, $1/(1-\sigma) \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$. Let p_σ be given by*

$$\frac{1}{p_\sigma} = \frac{1}{p} + \sigma. \tag{3.11}$$

Then, there exists a positive constant C depending only on p, q such that for any $(j,k,\ell) \in \Lambda$, the inequality

$$\|P_{\tilde{X}_{j,k,\ell}} F\|_{L^p_x L^q_t} \leqslant C 2^{-j\sigma} \|F\|_{L^{p_\sigma}_x L^q_t} \tag{3.12}$$

holds for any $F \in L^{p_\sigma}_x L^q_t$.

See [22, Proposition 2.3] for the proof of Proposition 3.2. Note that since P is a frequency cutoff which is smooth only in τ-direction, we are not free from a small *loss* in the exponent of x.

Let T be a operator defined by

$$(TF)(t,x) = \sum_{(j,k,\ell)\in\Lambda} P_{\tilde{A}_{j,k,\ell}} F(j,k,t,x) \tag{3.13}$$

for function $F = F(j,k,t,x)$, where $\tilde{A}_{j,k,\ell}$ is given by (3.9).

The Plancherel identity and the almost orthogonality (Proposition 3.1 (3.10)) imply

$$\|TF\|_{L^2_x L^2_t} \leqslant C \|F\|_{\dot{\ell}^2_j \ell^2_k L^2_x L^2_t}. \tag{3.14}$$

On the other hand, the triangle inequality and the boundedness for the Fourier multipliers $P_{\tilde{A}_{j,k,\ell}}$ (Proposition 3.2) yield

$$\|TF\|_{L^P_x L^Q_t} \leqslant C \|2^{-2j\theta\sigma} F\|_{\dot{\ell}^1_j \ell^1_k L^{P_\sigma}_x L^Q_t}, \tag{3.15}$$

where

$$\left(\theta, \frac{1}{P}, \frac{1}{Q}, \frac{1}{P_\sigma}\right) = \begin{cases} \left(\frac{p}{p-4}, 0, \frac{2(p-q)}{q(p-4)}, \frac{2p}{p-4}\sigma\right), & \text{if } p > q, \\ \left(\frac{q}{q-4}, \frac{2(q-p)}{p(q-4)}, 0, \frac{2(q-p)}{p(q-4)} + \frac{2q}{q-4}\sigma\right), & \text{if } p < q. \end{cases}$$

Interpolating (3.14) and (3.15), we obtain

$$\begin{aligned}(3.16)\ \|I_2\|_{L_x^{\frac{p}{2}} L_t^{\frac{q}{2}}} &\leqslant 2\Big\| \sum_{\tau_k^j \in \mathcal{D}} \sum_{\tau_\ell^j : \tau_\ell^j \sim \tau_k^j} |\partial_x|^s e^{-t\partial_x^3} f_{\tau_k^j} \overline{|\partial_x|^s e^{-t\partial_x^3} f_{\tau_\ell^j}} \Big\|_{L_x^{\frac{p}{2}} L_t^{\frac{q}{2}}} \\ &\leqslant C\bigg(\sum_{\tau_k^j \in \mathcal{D}} \sum_{\tau_\ell^j : \tau_\ell^j \sim \tau_k^j} |\tau_k^j|^{\delta\sigma} \| |\partial_x|^s e^{-t\partial_x^3} f_{\tau_k^j} \overline{|\partial_x|^s e^{-t\partial_x^3} f_{\tau_\ell^j}} \|^{\frac{\delta}{2}}_{L_x^{\frac{p_\sigma}{2}} L_t^{\frac{q}{2}}} \bigg)^{\frac{2}{\delta}},\end{aligned}$$

where the exponent p_σ is given by (3.11) and δ is given in Theorem 2.4. We have the similar inequality for I_1.

Step 4. By (3.1), (3.3) and (3.16), to show (2.3) it suffices to evaluate the right hand side of (3.16). We consider the case $p < q$ only since the case $p > q$ being similar. To evaluate the right hand side of (3.16), we consider an analytic family of linear operators $\{T_z^{j,k,\ell}\}_{z\in\mathbb{C}}$:

$$T_z^{j,k,\ell} : g \mapsto |\partial_x|^z e^{-t\partial_x^3} \mathcal{F}^{-1}[\mathbf{1}_{\tau_k^j} g] \cdot \overline{|\partial_x|^s e^{-t\partial_x^3} f_{\tau_\ell^j}}, \tag{3.17}$$

where $f_{\tau_\ell^j} \in |\partial_x|^{-\sigma} \hat{L}^{p_\sigma/2}$ is fixed. Employing the argument by [21, Proposition B.1] which is used to show Theorem 2.3, we obtain

$$\|T^{j,k,\ell}_{\frac{1}{p_\sigma}+i\gamma} g\|_{L_{t,x}^{\frac{p_\sigma}{2}}} \leqslant C \operatorname{dist}(0,\tau_\ell^j)^{s-\frac{1}{p_\sigma}-\sigma} |\tau_k^j|^{-\frac{2}{p_\sigma}} \|g\|_{L_\xi^{(\frac{p_\sigma}{2})'}} \| |\xi|^\sigma \hat{f}_{\tau_\ell^j} \|_{L_\xi^{(\frac{p_\sigma}{2})'}} \tag{3.18}$$

for any $\gamma \in \mathbb{R}$, where C is independent of γ. The linear Strichartz estimate for the Airy equation in $\hat{L}^p$ [20, Proposition 2.1] yields

$$\|T^{j,k,\ell}_{-\frac{1}{p_\sigma}+i\gamma} g\|_{L_x^{\frac{p_\sigma}{2}} L_t^\infty} \leqslant C \operatorname{dist}(0,\tau_\ell^j)^{s+\frac{1}{p_\sigma}-\sigma} \|g\|_{L_\xi^{(\frac{p_\sigma}{2})'}} \| |\xi|^\sigma \hat{f}_{\tau_\ell^j} \|_{L_\xi^{(\frac{p_\sigma}{2})'}} \tag{3.19}$$

for any $\gamma \in \mathbb{R}$, where C is independent of γ. Combining the Stein interpolation for the mixed norm (see [2, Section 7, Theorem 1]) with (3.18) and (3.19), we obtain

$$\|T^{j,k,\ell}_{s-\sigma} g\|_{L_x^{\frac{p_\sigma}{2}} L_t^{\frac{q}{2}}} \leqslant C|\tau_k^j|^{-\frac{2}{q}} \|g\|_{L_\xi^{(\frac{p_\sigma}{2})'}} \| |\xi|^\sigma \hat{f}_{\tau_\ell^j} \|_{L_\xi^{(\frac{p_\sigma}{2})'}}. \tag{3.20}$$

Collecting (3.16) and (3.20) with $g(\xi) = |\xi|^\sigma \hat{f}_{\tau_k^j}(\xi)$, we obtain

$$\|I_2\|_{L_x^{\frac{p}{2}} L_t^{\frac{q}{2}}} \leqslant C \| |\partial_x|^\sigma f\|^2_{\hat{M}^\beta_{\gamma,\delta}}. \tag{3.21}$$

In a similar way, we have the desired inequality for I_1. This completes the proof of Theorem 2.4 (2.3).

§ 4. Applications

§ 4.1. Applications to well-posedness of (gKdV)

In this subsection, we briefly recall the well-posedness of (gKdV) in space critical $\hat{M}^\beta_{\sigma,\delta}$ spaces.

To begin with, let us discuss the local well-posedness in $\hat{L}^\alpha$. Once we have Strichartz estimate in Theorem 1.2, we also obtain the following inhomogeneous estimates by a standard duality argument involving the Christ-Kiselev lemma (see [6, 24]).

Proposition 4.1 (Inhomogeneous Strichartz estimates [20]). *Let $4/3 < \alpha < 4$ and let (p_j, q_j) $(j = 1, 2)$ satisfy*

$$0 \leqslant \frac{1}{p_j} < \frac{1}{4}, \qquad 0 \leqslant \frac{1}{q_j} < \frac{1}{2} - \frac{1}{p_j}.$$

Then, the inequalities

$$\left\| \int_0^t e^{-(t-t')\partial_x^3} F(t')dt' \right\|_{L_t^\infty(I;\hat{L}_x^\alpha)} \leqslant C_1 \||\partial_x|^{-s_2} F\|_{L_x^{p_2'} L_t^{q_2'}(I)}, \tag{4.1}$$

and

$$\left\| |\partial_x|^{s_1} \int_0^t e^{-(t-t')\partial_x^3} F(t')dt' \right\|_{L_x^{p_1}(\mathbb{R};L_t^{q_1}(I))} \leqslant C_2 \||\partial_x|^{-s_2} F\|_{L_x^{p_2'}(\mathbb{R};L_t^{q_2'}(I))} \tag{4.2}$$

hold for any F satisfying $|D_x|^{-s_2} F \in L_x^{p_2'} L_t^{q_2'}$, where

$$\frac{1}{\alpha} = \frac{2}{p_1} + \frac{1}{q_1}, \quad s_1 = -\frac{1}{p_1} + \frac{2}{q_1}$$

and

$$\frac{1}{\alpha'} = \frac{2}{p_2} + \frac{1}{q_2}, \quad s_2 = -\frac{1}{p_2} + \frac{2}{q_2},$$

where the constant C_1 depends on α, s_1 and I, and the constant C_2 depends on α, s_1, s_2 and I.

By means of the estimate (4.2), we obtain the following version of the well-posedness.

Theorem 4.2. *Let $8/5 < \alpha < 10/3$. There exists $\delta_0 > 0$ such that if a function $u_0 \in \mathcal{S}'$, an interval I, and a time $t_0 \in I$ satisfy*

$$\delta := \left\| e^{-(t-t_0)\partial_x^3} u_0 \right\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R};L_t^{5\alpha}(I))} + \left\| |\partial_x|^{\frac{3}{4}-\frac{1}{2\alpha}} e^{-(t-t_0)\partial_x^3} u_0 \right\|_{L_x^{\frac{20\alpha}{10-3\alpha}}(\mathbb{R};L_t^{\frac{10}{3}}(I))} \leqslant \delta_0$$

then there exists a unique function $u(t,x)$ which satisfies

$$\|u\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R};L_t^{5\alpha}(I))} + \left\| |\partial_x|^{\frac{3}{4}-\frac{1}{2\alpha}} u \right\|_{L_x^{\frac{20\alpha}{10-3\alpha}}(\mathbb{R};L_t^{\frac{10}{3}}(I))} \leqslant 2\delta$$

and solves the equation

$$u(t) = e^{-(t-t_0)\partial_x^3} u_0 + \int_{t_0}^t e^{-(t-s)\partial_x^3} \partial_x(|u|^{2\alpha} u)(s) ds$$

in the $L_x^{\frac{5\alpha}{2}}(\mathbb{R}; L_t^{5\alpha}(I)) \cap L_x^{\frac{20\alpha}{10-3\alpha}}(\mathbb{R}; L_t^{\frac{10}{3}}(I))$ sense.

Proof. We have the following estimates as special cases of (4.2) for $8/5 < \alpha < 10/3$:

$$\left\| \int_0^t e^{-(t-s)\partial_x^3} \partial_x(|u|^{2\alpha}u)(s)ds \right\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R};L_t^{5\alpha}(I))} \lesssim \| |\partial_x|^{\frac{3}{4}-\frac{1}{2\alpha}}(|u|^{2\alpha}u)\|_{L_x^{\frac{20\alpha}{13\alpha+10}}(\mathbb{R};L_t^{\frac{10}{7}}(I))}$$

and

$$\left\| |\partial_x|^{\frac{3}{4}-\frac{1}{2\alpha}} \int_0^t e^{-(t-s)\partial_x^3} \partial_x(|u|^{2\alpha}u)(s)ds \right\|_{L_x^{\frac{20\alpha}{10-3\alpha}}(\mathbb{R};L_t^{\frac{10}{3}}(I))} \lesssim \| |\partial_x|^{\frac{3}{4}-\frac{1}{2\alpha}}(|u|^{2\alpha}u)\|_{L_x^{\frac{20\alpha}{13\alpha+10}}(\mathbb{R};L_t^{\frac{10}{7}}(I))}.$$

One also has the following nonlinear estimate (see [14, 20]):

$$\| |\partial_x|^{\frac{3}{4}-\frac{1}{2\alpha}}(|u|^{2\alpha}u)\|_{L_x^{\frac{20\alpha}{13\alpha+10}}(\mathbb{R};L_t^{\frac{10}{7}}(I))} \lesssim \|u\|^{2\alpha}_{L_x^{\frac{5\alpha}{2}}(\mathbb{R};L_t^{5\alpha}(I))} \left\| |\partial_x|^{\frac{3}{4}-\frac{1}{2\alpha}} u \right\|_{L_x^{\frac{20\alpha}{10-3\alpha}}(\mathbb{R};L_t^{\frac{10}{3}}(I))}. \tag{4.3}$$

From these estimates, we obtain the result by the standard contraction mapping principle. □

This theorem is essentially due to [20, Lemma 4.1]. The difference is that u_0 does not necessarily belong to $\hat{L}^\alpha$. By a further argument, we obtain standard results in perturbative argument in this frame work such as criterion for blowup and scattering, long time stability. For more detail, see [20, Section 4] and [21, Section 3]. By an another use of (1.2), we obtain the local-wellposedness in $\hat{L}^\alpha$.

Theorem 4.3 ([20]). *The initial value problem for* (gKdV) *is locally well-posed in $\hat{L}^\alpha$ if $8/5 < \alpha < 10/3$.*

Proof. We deduce from Theorem 1.2 that

$$\left\| e^{-t\partial_x^3} u_0 \right\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R};L_t^{5\alpha}(\mathbb{R}))} \lesssim \|u_0\|_{\hat{L}^\alpha}$$

and

$$\left\| |\partial_x|^{\frac{3}{4}-\frac{1}{2\alpha}} e^{-t\partial_x^3} u_0 \right\|_{L_x^{\frac{20\alpha}{10-3\alpha}}(\mathbb{R};L_t^{\frac{10}{3}}(\mathbb{R}))} \lesssim \|u_0\|_{\hat{L}^\alpha}.$$

Hence, we can choose $I \ni 0$ so that the assumption of Theorem 4.2 is satisfied. By using (4.1), one sees that $u \in C(I; \hat{L}^\alpha)$. □

As an application of the refinement of Strichartz' estimates, we show the well-posedness of (gKdV) in a scale critical $\hat{M}^\beta_{\gamma,\delta}$-type space. Notice that, as seen in the proof of Theorem 4.3, one crucial step of the well-posedness is to estimate space time norms by means of Strichartz' estimate. In view of the nonlinear estimate (4.3), it is natural that we need to handle a scale-critical spacetime norm without any differential, i.e., the norm

$$\|f\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R};L_t^{5\alpha}(I))}. \tag{4.4}$$

Remark that this norm is not diagonal. So we need a non-diagonal refinement to obtain the local well-posedness in a scale critical $\hat{M}^{\beta}_{\gamma,\delta}$-type space.

Assumption 4.4. Let $5/3 < \alpha \leqslant 20/9$ and $0 < \sigma \leqslant \min(3/5 - 1/\alpha, 1/4 - 2/(5\alpha))$. Define β by $1/\beta = 1/\alpha + \sigma$. Let γ and δ satisfy

$$\frac{4}{5\alpha} + 2\sigma \leqslant \frac{1}{\gamma} < \frac{1}{\beta}, \quad \frac{1}{2} - \frac{1}{5\alpha} \leqslant \frac{1}{\delta} < \frac{1}{\beta'}.$$

Under the above assumption, we have the following.

Theorem 4.5 (Local well-posedness in $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{\gamma,\delta}$)**.** *Suppose α, σ, β, γ. and δ satisfy Assumption 4.4. Then, the initial value problem* (gKdV) *is locally well-posed in $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{\gamma,\delta}$. More precisely, for any $|\partial_x|^{\sigma}u_0 \in \hat{M}^{\beta}_{\gamma,\delta}(\mathbb{R})$, there exist an interval $I = I(u_0)$ and a unique solution to* (gKdV) *satisfying*

$$u \in C(I; |\partial_x|^{-\sigma}\hat{M}^{\beta}_{\gamma,\delta}(\mathbb{R})) \cap L_x^{\frac{5\alpha}{2}}(\mathbb{R}; L_t^{5\alpha}(I)) \cap |\partial_x|^{-\frac{1}{3\beta}-\sigma} L^{3\beta}_{t,x}(I \times \mathbb{R}). \tag{4.5}$$

For any compact subinterval $I' \subset I$, there exists a neighborhood V of u_0 in $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{\gamma,\delta}(\mathbb{R})$ such that the map $u_0 \mapsto u$ from V into the class defined by (4.5) *with I' instead of I is Lipschitz continuous. The solution satisfies $u(t) - e^{-(t-t_0)\partial_x^3}u(t_0) \in C(I; \hat{L}^{\alpha} \cap |\partial_x|^{-\sigma}\hat{L}^{\beta})$ for any $t_0 \in I$.*

The strategy of the proof is the same as in Theorems 4.2 and 4.3. It seems that we need (4.4) to close the estimate. However, we have several options for the other norm. So, we choose $|\partial_x|^{-\frac{1}{3\beta}-\sigma}L^{3\beta}_{t,x}(I \times \mathbb{R})$ and obtain a well-posedness result similar to Theorem 4.2. This is the crucial step of the proof. The rest of the proof is similar to the proof of Theorem 4.3. We use the refined Strichartz estimate (Theorem 2.4). Remark that, in light of (4.1), the inhomogeneous term belongs to $C(I; |\partial_x|^{-\sigma}\hat{L}^{\beta})$. See [22, Theorems 1.7 and 1.8] for the detail of the proof.

By a standard argument, we also have small data scattering type result in this setting.

Theorem 4.6 (Small data scattering in $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{\gamma,\delta}$)**.** *Suppose α, σ, β, γ, and δ satisfy Assumption 4.4. Then, there exists $\varepsilon_0 > 0$ such that if $|\partial_x|^{\sigma}u_0 \in \hat{M}^{\beta}_{\gamma,\delta}(\mathbb{R})$ satisfies $\||\partial_x|^{\sigma}u_0\|_{\hat{M}^{\beta}_{\gamma,\delta}} \leqslant \varepsilon_0$, then the solution $u(t)$ to* (gKdV) *given in Theorem 4.5 is global in time and scatters for both time directions. Moreover,*

$$\||\partial_x|^{\sigma}u\|_{L_t^{\infty}(\mathbb{R};\hat{M}^{\beta}_{\gamma,\delta})} + \|u\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R};L_t^{5\alpha}(\mathbb{R}))} \leqslant 2\||\partial_x|^{\sigma}u_0\|_{\hat{M}^{\beta}_{\gamma,\delta}}. \tag{4.6}$$

§ 4.2. Application to the existence of a special non-scattering solutions I

We next consider the application of the refined Strichartz estimates to construction of a minimal non-scattering solution to mass-subcritical generalized KdV equation (gKdV). This is based on a concentration compactness argument initiated by Kenig and Merle [12].

Here, let us see that this argument works if we take $\gamma = 2$, that is, in the framework of $\hat{M}^{\alpha}_{2,\delta}$ or $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$. We introduce the following three deformations, groups of isometric transforms, associated with the spaces $\hat{M}^{\alpha}_{\gamma,\delta}$ or $|\partial_x|^{-\sigma}\hat{M}^{\alpha}_{\gamma,\delta}$:

- Translation in Physical side: $(T(y)f)(x) := f(x-y),\ y \in \mathbb{R}$.
- Airy flow: $(A(t)f)(x) = (e^{-t\partial_x^3}f)(x),\ t \in \mathbb{R}$.
- Dilation (scaling): $(D(h)f)(x) = h^{\alpha}f(hx),\ h \in 2^{\mathbb{Z}}$.

These also relates to the symmetries of the equation (gKdV). Remark that the orbit of a fixed nonzero function with respect to these deformations forms a non-compact bounded set in $\hat{M}^{\alpha}_{\gamma,\delta}$ or $|\partial_x|^{-\sigma}\hat{M}^{\alpha}_{\gamma,\delta}$.

One of the key tools for this argument is a (linear) profile decomposition. The rough statement is as follows: When a sense of "boundedness" and a sense of "smallness" are chosen, we specify the corresponding deformation and give a procedure to decompose (up to a subsequence) a sequence bounded in the chosen sense into a sum of mutually asymptotically orthogonal profiles, each one of which is an element of the orbit of some (fixed) function with respect to the specified deformation, and the remainder which satisfies the chosen smallness in a suitable sense.

The crucial step of the proof of the linear profile decomposition is a control of the vanishing scenario, namely, to prove that if a bounded sequence does not satisfy the designated smallness then the sequence has a nonzero weak limit along a subsequence modulo the deformation. Remark that the deformation in a profile decomposition result is determined in this step. In recent versions used in the Kenig-Merle theory, the smallness is with respect to a spacetime norm, which is often called a *scattering norm*, of a linear propagation of the function. Refined Strichartz' estimates are suitable for this purpose and have been used for the analysis of mass-critical case [1, 4, 5, 7, 15, 16, 23, 27]. We have a control result in $\hat{M}^{\alpha}_{\gamma,\delta}$ or $|\partial_x|^{-\sigma}\hat{M}^{\alpha}_{\gamma,\delta}$ without the restriction $\gamma = 2$, or even in $\hat{L}^{\alpha}$, by our refined estimates in Theorems 2.3 and 2.4. However, in our setting the norm (4.4) plays a role of the scattering norm. To include the norm as the smallness of the decomposition, we need nondiagonal estimate.

Intuitively, the proof of the linear profile decomposition is done by the recurrence use of the above control result to the remainder term. However, to get situation better as the number of the detected profiles increase, we need a *decoupling equality.* For this part, we need the restriction $\gamma = 2$.

Let us be more precise about this fact. Consider the following sequence:

$$f_n = T(n)\phi_1 + \phi_2, \quad \phi_1, \phi_2 \in \mathcal{S}.$$

Let $p \in (1, \infty)$. By the Brezis-Lieb lemma, we have

$$\|f_n\|_{L^p}^p = \|\phi_1\|_{L^p}^p + \|\phi_2\|_{L^p}^p + o(1)$$

as $n \to \infty$ because f_n converges almost everywhere to its weak limit ϕ_2 as $n \to \infty$. This is what we mean by decoupling equality. However, we do not have decoupling equality in $\hat{L}^p$. Indeed, the transform is

$$\mathcal{F}f_n(\xi) = e^{in\xi}\mathcal{F}\phi_1(\xi) + \mathcal{F}\phi_2(\xi).$$

The sequence $\{\mathcal{F}f_n\}_n$ converges to $\mathcal{F}\phi_2$ weakly in $L^{p'}$ but not everywhere. Two parts "decouples" because of the phase oscillation. Hence, in general,

$$\|\mathcal{F}f_n\|_{L^{p'}}^{p'} = \|\mathcal{F}\phi_1\|_{L^{p'}}^{p'} + \|\mathcal{F}\phi_2\|_{L^{p'}}^{p'} + o(1)$$

holds only for $p = 2$. Indeed, this identity fails when $p' = 4$.

This is where we need $\gamma = 2$. Thanks to the L^2 structure, we have the decoupling equality in each dyadic interval. Summing up these decoupling equalities over dyadic intervals in ℓ^δ sense, one obtains a *decoupling inequality*. (Recall that $\delta > 2$.)

We close this subsection with introducing the precise statement of the profile decomposition given as an application of Theorem 2.3. Let us make the following assumption.

Assumption 4.7. We suppose Assumption 4.4 with $\gamma = 2$ and exclude the endpoint cases, i.e., let $5/3 < \alpha < 12/5$ and $\max(0, 1/2 - 1/\alpha) < \sigma < \min(3/5 - 1/\alpha, 1/4 - 2/(5\alpha))$. Define $\beta \in (5/3, 2)$ by $1/\beta = 1/\alpha + \sigma$ and let $1/\delta \in (1/2 - 1/(5\alpha), 1/\beta')$.

We also introduce the notion of the orthogonality of the two families of deformations. Here we consider the deformation of the form $D(h)A(s)T(y)$ with $h \in 2^{\mathbb{Z}}$ and $s, y \in \mathbb{R}$. Hence, the notion is described in terms of the corresponding families of the parameters.

Definition 4.8. We say two families of paraemters $\{(h_n^1, s_n^1, y_n^1)\}_n \subset 2^{\mathbb{Z}} \times \mathbb{R}^2$ and $\{(h_n^2, s_n^2, y_n^2)\}_n \subset 2^{\mathbb{Z}} \times \mathbb{R}^2$ are *orthogonal* if

$$\lim_{n\to\infty} \left(\left| \log \frac{h_n^1}{h_n^2} \right| + \left| s_n^1 - \left(\frac{h_n^1}{h_n^2} \right)^3 s_n^2 \right| + \left| y_n^1 - \frac{h_n^1}{h_n^2} y_n^2 \right| \right) = +\infty. \tag{4.7}$$

Theorem 4.9 (Linear profile decomposition in $|\partial_x|^{-\sigma}\hat{M}_{2,\delta}^\beta$)**.** *Suppose that α, σ, β, γ, and δ satisfy Assumption 4.7. Let $\{u_n\}_n$ be a bounded sequence in $|\partial_x|^{-\sigma}\hat{M}_{2,\delta}^\beta$.*

Then, there exist $\psi^j \in |\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$, $r^j_n \in |\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$, and pairwise orthogonal families of parameters $\{(h^j_n, s^j_n, y^j_n)\}_n$ ($j = 1, 2, \dots$) such that, extracting a subsequence in n,

$$u_n = \sum_{j=1}^{J} D(h^j_n)A(s^j_n)T(y^j_n)\psi^j + r^J_n \tag{4.8}$$

for all $n, J \geqslant 1$ and

$$\lim_{J\to\infty} \varlimsup_{n\to\infty} \left(\left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} r^J_n \right\|_{L^{3\alpha}_{t,x}(\mathbb{R}\times\mathbb{R})} + \left\| e^{-t\partial_x^3} r^J_n \right\|_{L_t^{\frac{5\alpha}{2}} L_x^{5\alpha}(\mathbb{R}\times\mathbb{R})} \right) = 0 \tag{4.9}$$

Moreover, a decoupling inequality

$$\varlimsup_{n\to\infty} \| |\partial_x|^{\sigma} u_n \|^{\delta}_{\hat{M}^{\beta}_{2,\delta}} \geqslant \sum_{j=1}^{J} \| |\partial_x|^{\sigma} \psi^j \|^{\delta}_{\hat{M}^{\beta}_{2,\delta}} + \varlimsup_{n\to\infty} \| r^J_n \|^{\delta}_{\hat{M}^{\beta}_{2,\delta}} \tag{4.10}$$

holds for all $J \geqslant 1$. Furthermore, if u_n is real-valued then so are ψ^j and r^J_n.

§ 4.3. Application to the existence of a special non-scattering solutions II

In the rest of this section, we briefly summarize what we can obtain from Theorem 4.9 without any proof. For the details, see [22]. In this subsection. we suppose Assumption 4.7. A solution always implies a $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$-solution given in Theorem 4.5.

Let us first introduce notations. For a solution $u(t)$ on I, take $t_0 \in I$ and set

$$\begin{aligned} T_{\max} &:= \sup\{T > t_0 \mid u(t) \text{ can be extended to a solution on } [t_0, T).\}, \\ T_{\min} &:= \sup\{T > -t_0 \mid u(t) \text{ can be extended to a solution on } (-T, t_0].\}, \\ I_{\max} &= I_{\max}(u) := (-T_{\min}, T_{\max}). \end{aligned}$$

Definition 4.10 (Scattering). We say a solution $u(t)$ scatters forward in time (resp. backward in time) if $T_{\min} = \infty$ (resp. $T_{\max} = \infty$) and if $|\partial_x|^{\sigma} e^{t\partial_x^3} u(t)$ converges in $\hat{M}^{\beta}_{2,\delta}$ as $t \to \infty$ (resp. $t \to -\infty$).

We consider the following two minimization problems.

$$E_1 := \inf \left\{ \inf_{t\in I_{\max}} \| |\partial_x|^{\sigma} u(t) \|_{\hat{M}^{\beta}_{2,\delta}} \;\middle|\; \begin{array}{l} u(t) \text{ is a solution to (gKdV) that} \\ \text{does not scatter forward in time.} \end{array} \right\}.$$

Theorem 4.6 is represented as $E_1 > 0$. Remark that it holds that

$$E_1 = \inf \left\{ \| |\partial_x|^{\sigma} u(0) \|_{\hat{M}^{\beta}_{2,\delta}} \;\middle|\; \begin{array}{l} u(t) \text{ is a solution to (gKdV) that does} \\ \text{not scatter forward in time, } 0 \in I_{\max}(u). \end{array} \right\}.$$

by the time translation symmetry. Further, one sees that E_1 is the supremum of the number ε_0 for which Theorem 4.6 is true.

We also introduce another infimum value.

$$E_2 := \inf \left\{ \varlimsup_{t\uparrow T_{\max}} \||\partial_x|^\sigma u(t)\|_{\hat{M}^\beta_{2,\delta}} \,\middle|\, \begin{array}{l} u(t) \text{ is a solution to (gKdV) that} \\ \text{does not scatter forward in time.} \end{array} \right\}.$$

By definition, $E_1 \leqslant E_2 \leqslant \||\partial_x|^\sigma Q\|_{\hat{M}^\beta_{2,\delta}}$ for the focusing case (i.e. $\mu < 0$), where $Q(x)$ is a (unique) positive even solution of $-Q'' + Q = Q^{2\alpha+1}$. For another characterization of this quantity, see Remark below. The goal is to determine the explicit value of E_j $(j = 1, 2)$. Here, we will show that existence of minimizers to both E_1 and E_2, which would be a important step.

In what follows, we consider the focusing case $\mu < 0$ only. However, the focusing assumption is used only for assuring E_j are finite. Our analysis work also in the defocusing case $\mu > 0$ if we assume E_j are finite.

Theorem 4.11 (Analysis of E_1). *Suppose that Assumption 4.7 is satisfied. Then,* $0 < E_1 \leqslant c_\alpha \||\partial_x|^\sigma Q\|_{\hat{M}^\beta_{2,\delta}}$, *where* $c_\alpha = \min(1, (\alpha/2)^{1/2\alpha})$. *Furthermore, there exists a minimizer* $u_1(t)$ *to* E_1 *in the following sense:* $u_1(t)$ *is a solution to* (gKdV) *with maximal interval* $I_{\max}(u_1) \ni 0$ *and*

1. *$u_1(t)$ does not scatter forward in time;*

2. *$u_1(t)$ attains E_1 in such a sense that either one of the following two properties holds;*

   (a) $\||\partial_x|^\sigma u_1(0)\|_{\hat{M}^\beta_{2,\delta}} = E_1$;

   (b) $u_1(t)$ *scatters backward in time and* $u_{1,-} := \lim_{t\to-\infty} e^{t\partial_x^3} u_1(t)$ *satisfies* $\||\partial_x|^\sigma u_{1,-}\|_{\hat{M}^\beta_{2,\delta}} = E_1$.

Theorem 4.12 (Analysis of E_2). *Suppose that Assumption 4.7 is satisfied. Then,* $E_1 \leqslant E_2 \leqslant \||\partial_x|^\sigma Q\|_{\hat{M}^\beta_{2,\delta}}$. *Furthermore, there exists a minimizer* $u_2(t)$ *to* E_2 *in the following sense:* $u_2(t)$ *is a solution to* (gKdV) *with maximal interval* $I_{\max}(u_2) \ni 0$ *and*

1. *$u_2(t)$ does not scatter forward and backward in time;*

2. *Three quantities*

$$\sup_{t\in\mathbb{R}} \||\partial_x|^\sigma u_2(t)\|_{\hat{M}^\beta_{2,\delta}}, \quad \varlimsup_{t\uparrow T_{\max}} \||\partial_x|^\sigma u_2(t)\|_{\hat{M}^\beta_{2,\delta}}, \quad \varlimsup_{t\downarrow T_{\min}} \||\partial_x|^\sigma u_2(t)\|_{\hat{M}^\beta_{2,\delta}}$$

are equal to E_2.

3. $u_2(t)$ *is precompact modulo symmetries, i.e., there exist a scale function* $N(t) : I_{\max} \to \mathbb{R}_+$ *and a space center* $y(t) : I_{\max} \to \mathbb{R}$ *such that the set*

$$\{(D(N(t))T(y(t)))^{-1}u_2(t) \mid t \in I_{\max}\} \subset |\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$$

is precompact.

Remark 4. We give another characterization of E_2. For $E \geqslant 0$, we define

$$\mathcal{L}(E) := \sup \left\{ \|u\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R};L_t^{5\alpha}(I))} \;\middle|\; \begin{array}{l} u(t) \in C(I; |\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}) \text{ is a solution} \\ \text{to (gKdV) on a compact interval } I \\ \text{such that } \max\limits_{t\in I} \||\partial_x|^{\sigma}u(t)\|_{\hat{M}^{\beta}_{2,\delta}} \leqslant E \end{array} \right\}.$$

Remark that $\mathcal{L} : [0,\infty) \to [0,\infty]$ is non-decreasing. Then, $E_2 = \sup\{E \mid \mathcal{L}(E) < \infty\} = \inf\{E \mid \mathcal{L}(E) = \infty\}$ holds.

To prove Theorems 4.11 and 4.12, we establish the linear profile decomposition in $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$. Note that Theorem 2.4 is used to control the vanishing of minimizing sequence. See [22, Theorem 4.1] for the detail of the proofs of Theorems 4.11.

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References

[1] Bégout P. and Vargas A., *Mass concentration phenomena for the* L^2*-critical nonlinear Schrödinger equation.* Trans. Amer. Math. Soc. **359** (2007), 5257–5282.

[2] Benedek A. and Panzone R., *The space* L^p*, with mixed norm*, Duke Math. J. **28** (1961) 301–324.

[3] Bourgain J., *On the restriction and multiplier problems in* $\mathbb{R}^3$. Geometric aspects of functional analysis (1989–90), Lecture Notes in Math., **1469**, Springer, Berlin, (1991), 179–191.

[4] Bourgain J., *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity.* Internat. Math. Res. Notices **1998** (1998), 253–283.

[5] Carles R., and Keraani S., *On the role of quadratic oscillations in nonlinear Schrödinger equations II. The* L^2*-critical case.* Trans. Amer. Math. Soc. **359** (2007), 33–62.

[6] Christ M. and Kiselev A., *Maximal functions associated to filtrations*, J. Funct. Anal. **179** (2001) 409–425.

[7] Dodson. B., *Global well-posedness and scattering for the defocusing, mass-critical generalized KdV equation.* Ann. PDE **3** (2017), Article no.5.

[8] Farah L.G., Linares F., Pastor A. and Visciglia N. *Large data scattering for the defocusing supercritical generalized KdV equation*, Comm. Partial Differential Equations **43** (2018), 118–157.

[9] Fefferman C., *Inequalities for strongly singular convolution operators.* Acta Math. **124** (1970) 9–36.

[10] Grünrock A., *An improved local well-posedness result for the modified KdV equation.* Int. Math. Res. Not. **2004** (2004), 3287–3308.

[11] Kato T., *On the Cauchy problem for the (generalized) KdV equation.* Advances in Math. Supplementary studies, Studies in Applied Mathematics **8** (1983), 93–128.

[12] Kenig C.E. and Merle F., *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case.* Invent. Math. **166** (2006), 645–675.

[13] Kenig C.E., Ponce G. and Vega L., *Oscillatory integrals and regularity of dispersive equations.* Indiana Univ.math J. **40** (1991), 33–69.

[14] Kenig C.E., Ponce G. and Vega L., *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle.* Comm. Pure Appl. Math. **46** (1993), 527–620.

[15] Kenig C.E., Ponce G. and Vega L., *On the concentration of blow up solutions for the generalized KdV equation critical in L^2.* Nonlinear wave equations (Providence, RI 1998), Contemp. Math. **263**, Amer. Math. Soc., Providence, RI (2000), 131–156.

[16] Killip R, Kwon S., Shao S. and Vişan M., *On the mass-critical generalized KdV equation.* Discrete Contin. Dyn. Syst. **32** (2012), 191–221.

[17] Koch H. and Marzuola J.L., *Small data scattering and soliton stability in $\dot{H}^{-\frac{1}{6}}$ for the quartic KdV equation.* Anal. PDE **5** (2012), 145–198.

[18] Kwon S. and Roy T., *Bilinear local smoothing estimate for Airy equation.* Differential Integral Equations **25** (2012), 75–83.

[19] Masaki S., *Two minimization problems on non-scattering solutions to mass-subcritical nonlinear Schrödinger equation.* preprint available at arXiv:1605.09234 (2016).

[20] Masaki S. and Segata J., *On well-posedness of generalized Korteweg-de Vries equation in scale critical $\hat{L}^r$ space.* Anal. and PDE. **9** (2016), 699–725.

[21] Masaki S. and Segata J., *Existence of a minimal non-scattering solution to the mass-subcritical generalized Korteweg-de Vries equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **35** (2018), 283–326.

[22] Masaki S. and Segata J., *Refinement of Strichartz estimates for Airy Equation in nondiagonal case and its application*, SIAM J. Math. Anal. **50** (2018), 2839–2866.

[23] Merle F. and Vega L., *Conpactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2D.* Internat. Math. Res. Notices **1998** (1998), 399-425.

[24] Molinet L. and Ribaud F., *Well-posedness results for the generalized Benjamin-Ono equation with small initial data.* J. Math. Pures Appl. **83** (2004), 277–311.

[25] Moyua, A., Vargas, A., and Vega, L., *Schrödinger maximal function and restriction properties of the Fourier transform.* Internat. Math. Res. Notices **1996** (1996), 793–815.

[26] Moyua, A., Vargas, A., and Vega, L., *Restriction theorems and maximal operators related to oscillatory integrals in $\mathbb{R}^3$.* Duke Math. J. **96** (1999), no. 3, 547–574.

[27] Shao S., *The linear profile decomposition for the Airy equation and the existence of maximizers for the Airy Strichartz inequality.* Anal. PDE **2** (2009), 83–117.

[28] Tao T., *Scattering for the quartic generalised Korteweg-de Vries equation.* J. Differential Equations **232** (2007), 623–651.

[29] Tao T., Vargas A., and Vega L., *A bilinear approach to the restriction and Kakeya conjectures.* J. Amer. Math. Soc. **11** (1998), 967–1000.

[30] Tomas P. A., *A restriction theorem for the Fourier transform.* Bull. Amer. Math. Soc. **81**

(1975), 477–478.

[31] Vargas A. and Vega L., *Global wellposedness for 1D non-linear Schrödinger equation for data with an infinite L^2 norm.* J. Math. Pures Appl. (9) **80** (2001), 1029–1044.